Matematička analiza 1 - 20. auditorna vježba - RJEŠENJA

Zadatak 1 (a)

$$\int \frac{dx}{x^3-3x-2} = \int \frac{dx}{(x+1)^2(x-2)} = \int \left(-\frac{1}{9(x+1)} - \frac{1}{3(x+1)^2} + \frac{1}{9(x-2)}\right) dx =$$
$$= -\frac{1}{9}\ln|x+1| + \frac{1}{3(x+1)} + \frac{1}{9}\ln|x-2| + C, \quad C \in \mathbb{R}.$$

(b)

$$\int \frac{6x^2-5x+2}{4x^3+x+5}dx = \int \frac{6x^2-5x+2}{(x+1)(4x^2-4x+5)}dx =$$
$$= \int \left(\frac{1}{x+1} + \frac{2x-3}{4x^2-4x+5}\right) dx = \int \left(\frac{1}{x+1} + \frac{1}{4}\frac{8x-4}{4x^2-4x+5} - \frac{2}{4x^2-4x+5}\right) dx =$$
$$= \ln|x+1| + \frac{1}{4}\ln|4x^2-4x+5| - \int \frac{2}{(2x-1)^2+4}dx =$$
$$= \ln|x+1| + \frac{1}{4}\ln|4x^2-4x+5| - \frac{1}{2}\text{arctg}(\frac{2x-1}{2}) + C, \quad C \in \mathbb{R}.$$

(c)

$$\int \frac{x^2+1}{x^4+x^2+1}dx = \int \frac{x^2+1}{(x^2+x+1)(x^2-x+1)}dx = \int \left(\frac{1}{2}\frac{1}{x^2-x+1} + \frac{1}{2}\frac{1}{x^2+x+1}\right) dx =$$
$$= \frac{1}{2}\int \left(\frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}} + \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\right) dx =$$
$$\frac{1}{2}\left(\frac{2}{\sqrt{3}}\text{arctg}(\frac{2x-1}{\sqrt{3}}) + \frac{2}{\sqrt{3}}\text{arctg}(\frac{2x+1}{\sqrt{3}}) + C\right) =$$
$$\frac{1}{\sqrt{3}}\left(\text{arctg}(\frac{2x-1}{\sqrt{3}}) + \text{arctg}(\frac{2x+1}{\sqrt{3}})\right) + C, \quad C \in \mathbb{R}.$$

Zadatak 2 (a)

$$\int \text{tg}(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = \left[\begin{smallmatrix} t=\cos(x) \\ dt=-\sin(x)dx \end{smallmatrix}\right] = -\int \frac{1}{t}dt =$$
$$= -\ln|t| + C = -\ln|\cos(x)| + C, \quad C \in \mathbb{R}.$$

(b)

$$\int \text{ctg}(x)dx = \int \frac{\cos(x)}{\sin(x)}dx = \left[\begin{smallmatrix} t=\sin(x) \\ dt=\cos(x)dx \end{smallmatrix}\right] = \int \frac{1}{t}dt =$$
$$= \ln|t| + C = \ln|\sin(x)| + C, \quad C \in \mathbb{R}.$$

(c)

$$\int \left(\sin(x)^5 + \cos(x)^5\right) dx = \int \sin(x)^5 dx + \int \cos(x)^5 dx =$$
$$= \int \left(1-\cos(x)^2\right)^2 \sin(x)dx + \int \left(1-\sin(x)^2\right)^2 \cos(x)dx = \left[\begin{smallmatrix} s=\cos(x), t=\sin(x) \\ ds=-\sin(x)dx, dt=\cos(x)dx \end{smallmatrix}\right] =$$
$$= -\int (1-s^2)^2 ds + \int (1-t^2)^2 dt = -\int \left(1-2s^2+s^4\right) ds + \int \left(1-2t^2+t^4\right) dt =$$
$$= -\left(s - \frac{2}{3}s^3 + \frac{1}{5}s^5\right) + \left(t - \frac{2}{3}t^3 + \frac{1}{5}t^5\right) + C =$$
$$= -\cos(x) + \frac{2}{3}\cos(x)^3 - \frac{1}{5}\cos(x)^5 + \sin(x) - \frac{2}{3}\sin(x)^3 + \frac{1}{5}\sin(x)^5 + C, \quad C \in \mathbb{R}.$$

(d)

$$\int \sin(x)^4\cos(x)^2dx = \int \sin(x)^4\cos(x)\cdot\cos(x)dx = \left[\begin{smallmatrix} du=\sin(x)^4\cos(x)dx,\ v=\cos(x) \\ u=\frac{1}{5}\sin(x)^5,\ dv=-\sin(x)dx \end{smallmatrix}\right] =$$

$$= \frac{1}{5}\sin(x)^5\cos(x) + \frac{1}{5}\int \sin(x)^6 dx = \frac{1}{5}\sin(x)^5\cos(x) + \frac{1}{5}\int \left(\frac{1-\cos(2x)}{2}\right)^3 dx =$$

$$= \frac{1}{5}\sin(x)^5\cos(x) + \frac{1}{5}\int \frac{1-3\cos(2x)+3\cos(2x)^2-\cos(2x)^3}{8}dx =$$

$$= \frac{1}{5}\sin(x)^5\cos(x) + \frac{1}{5}\int \frac{1-3\cos(2x)+\frac{3(1+\cos(4x))}{2}-(1-\sin(2x)^2)\cos(2x)}{8}dx =$$

$$= \frac{1}{5}\sin(x)^5\cos(x) + \frac{1}{5}\int \frac{5-8\cos(2x)+3\cos(4x)+2\sin(2x)^2\cos(2x)}{16}dx =$$

$$= \frac{1}{5}\sin(x)^5\cos(x) + \frac{1}{5}\left(\frac{5x}{16} - \frac{1}{4}\sin(2x) + \frac{3}{64}\sin(4x) + \frac{1}{48}\sin(2x)^3\right) + C, \quad C\in\mathbb{R}.$$

Zadatak 3 (a)

$$\int \frac{\sqrt{1-x^2}}{x^4}dx = \left[\begin{smallmatrix} x=\sin(t),\sqrt{1-x^2}=\cos(t) \\ dx=\cos(t)dt \end{smallmatrix}\right] = \int \frac{\cos(t)^2}{\sin(t)^4}dt =$$

$$= \int \operatorname{ctg}(t)^2\frac{dt}{\sin(t)^2} = -\frac{1}{3}\operatorname{ctg}(t)^3 + C = -\frac{(1-x^2)\sqrt{1-x^2}}{3x^3} + C, \quad C\in\mathbb{R}.$$

(b)

$$\int \frac{\sqrt{1-x^2}}{x^3}dx = \left[\begin{smallmatrix} t=\sqrt{1-x^2} \\ dt=-\frac{x}{\sqrt{1-x^2}}dx \end{smallmatrix}\right] = \int -\frac{t^2}{(1-t^2)^2}dt = \int -\frac{t^2}{(t-1)^2(t+1)^2}dt =$$

$$= \int \left(-\frac{1}{4(t-1)} - \frac{1}{4(t-1)^2} + \frac{1}{4(t+1)} - \frac{1}{4(t+1)^2}\right)dt =$$

$$= \frac{1}{4}\left(-\ln|t-1| + \frac{1}{t-1} + \ln|t+1| + \frac{1}{t+1}\right) + C =$$

$$= \frac{1}{4}\left(\ln|\frac{t+1}{t-1}| + \frac{2t}{t^2-1}\right) + C =$$

$$\frac{1}{4}\left(\ln|\frac{\sqrt{1-x^2}+1}{\sqrt{1-x^2}-1}| + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}^2-1}\right) + C =$$